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THE GENERALIZED CATTANEO PARTIAL SLIP PLANE CONTACT PROBLEM. I—THEORY†

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Abstract—The Cattaneo problem is considered for a general plane contact between elastically similar materials, i.e. a monotonically increasing tangential load, starting from zero, with normal loading held fixed. Instead of the classical argument on the displacement field in the stick zone of Cattaneo solution, we attack the problem implicitly from the governing integral equations in the stick zones. After discussing and solving the full-stick case, we move to the more realistic (for finite friction) case of partial slip. We show that, upon isolating the effect of full sliding, the equalities and inequalities governing the corrective solution for the *corrective* shearing tractions in the stick zone are *exactly* the same as those governing the solution of the normal contact problem with a lower load, but the same rotation as the actual one. This analogy permits us to deduce several general properties, and gives a general procedures for solving partial slip Cattaneo problems as frictionless normal indentation ones. Therefore, the general solutions for single, multiple and periodic contacts is given. A comprehensive set of explicit results is given in the part II of the paper.

1. INTRODUCTION

The well-known Cattaneo procedure (Cattaneo, 1938) has for a long time been the most widely used technique for solving a contact in the partial slip regime. Cattaneo attacked the problem of partial slip considering a constant normal component of the load P, and a monotonically increasing tangential component Q, starting from zero, for the general 3D Hertzian case of elliptical contact area. He used an argument involving the displacement field in the stick zone, where, because of symmetry and Newton's third law, the relative tangential displacements of the contacting bodies can be only a rigid body motion. In two subsequent papers (Cattaneo, 1947a, 1947b), the axisymmetric case for a fourth-order function is also treated. The simpler case for axisymmetric and elliptical Hertzian contacts was treated again several years later, apparently independently, by Mindlin (Mindlin, 1949), whereas some generalizations of the loading path were considered in Mindlin *et al.* (1952), and Mindlin and Dereciewicz (1953). Experimental evidence of the theory was provided also by Johnson (1955). Although sometimes the tangential problem is referred to as the Cattaneo–Mindlin problem, we shall refer in this paper to the *Cattaneo problem*, for clarity, as we reconsider the original problem considered by Cattaneo, namely the single monotonic application of tangential load starting from zero, whereas the *Mindlin problem* should refer, more appropriately, to the innovative part of Mindlin's work, namely the general loading path in the papers of 1952–1953.

For the classical basic case in plane geometry, namely the contact of cylinders (approximated by parabolas), several aspects of partial slip for elastically similar or dissimilar materials have already been worked out in detail, either analytically or numerically, including the effect of the presence of bulk stresses in one of the contacting bodies, and for the principal results the reader is referred to Hills *et al.* (1993, Chapter 4). Apart from this geometry, very few other cases of Cattaneo's partial slip plane problems have been solved to date. In practice, to the best of the author's knowledge, there is only the case of the wedge indenter, recently solved by (Truman *et al.*, 1995). This is probably because the Cattaneo procedure involves the explicit calculation of the displacements induced, whereas

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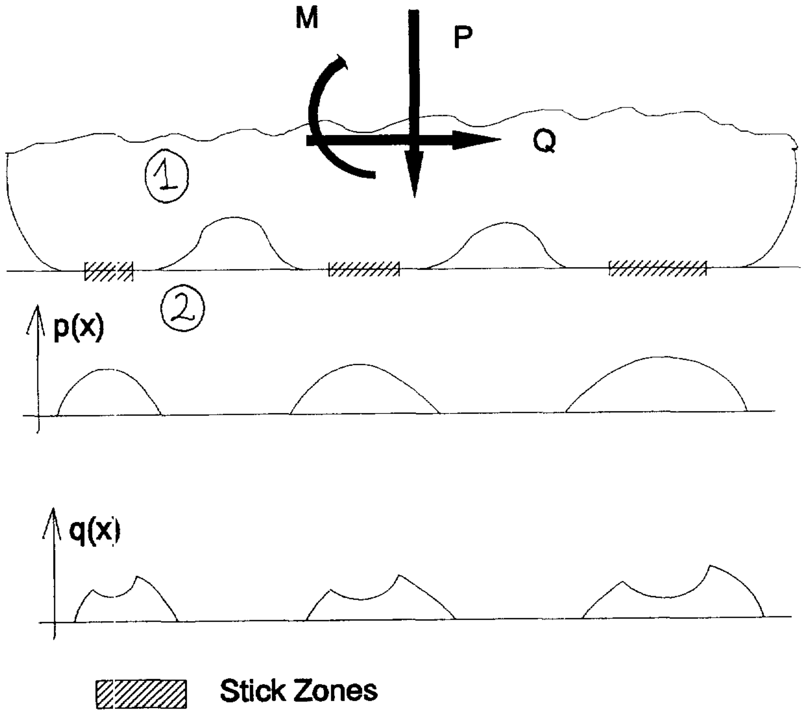

Fig. 1. General multiply-connected plane contact.

our method will completely avoid this step, formulating the problem directly in terms of integral equations. For plane contacts there is, in fact, *exact* correspondence (except from a constant) between tangential displacements due to a distribution of shearing tractions on the half-plane, and normal displacements due to a similar distribution of normal tractions. With this in mind, we will show an analogy between the case of partial slip for the Cattaneo problem, and the normal frictionless indentation, and subsequently obtain general properties. These properties will hold in very general cases—for example we do not require the contact area to be simply-connected.

Analytical results can then be obtained for all the geometries for which the normal frictionless contact can be solved analytically. We will recall general solutions of the contact problem, that translate immediately into solutions for a partial slip Cattaneo problem. In particular, general solutions in quadrature are given for the case of single contact (either symmetrical or non-symmetrical), multiple contact, and periodic contact. Regarding explicit solutions, part II of the paper gives details of the derivation of many closed form solutions for cases of engineering interest.

It is worth remarking that we consider, as the main result, the property that permits us to show that a Cattaneo partial slip contact problem can be interpreted as a superposition of normal contact ones, by using a full sliding component and a corrective problem where the normal load is reduced appropriately, but the relative rotation is fixed to the value given by the *actual* normal contact. This greatly simplifies the solution of the Cattaneo problem in general, including the numerical solution of cases where the analytical formula are cumbersome to treat.

2. FORMULATION

In order to keep the formulation as succinct as possible we will first write down the governing equations for the most general case. Figure 1 shows schematically an indent subject to a normal force, P, a moment M (clearly it is always possible to find a point of application of P such that $M = 0$, but we indicate with M the moment with respect to $x = 0$, and P is applied along the line $x = 0$). At the end of the normal loading phase, a tangential force Q is applied (further details will be given below) at $y = 0$. It is assumed that there are stick zones S_{stick}, and slip zones S_{slip}, wherein the shear tractions are strictly related to local normal pressure by Coulomb's friction law, as given below.

Define the function $h(x)$ as the amount of overlap if the bodies were allowed to interpenetrate each other freely, as

$$h(x) = T_y + Rx - [f_1(x) - f_2(x)] \tag{1}$$

where $f_1(x)$, $f_2(x)$ describe the profile of the upper and lower (1 and 2, respectively) contacting bodies, T_y and R are the normal and rotational components, respectively, of the rigid body motion that brings the two bodies into contact, given in the fixed coordinate system $x-y$. Compatibility of displacements in the normal direction gives a first equality (inequality) over the contact area S (outside S) in terms of the y-component of relative displacements $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$

$$u_y(x) = h(x), \quad x \in S \tag{2}$$

$$u_y(x) > h(x), \quad x \notin S \tag{3}$$

where the latter condition avoids interpenetration outside the contact area (note that the contact pressure can only be compressive, as for definition, tensile stresses cannot be sustained, so that $p(x) = \sigma_{yy}(x,0) < 0, x \in S$). Equation (2) can be written in terms of the traction distribution, by employing logarithmically singular integral equations, whereas inequality (3) becomes a non-singular integral inequality. However, as the displacements in contact problems for half-planes are logarithmically unbounded at infinity, it is usually preferred to work with displacement derivatives, writing

$$\frac{1}{A}h'(x) = \frac{1}{\pi}\int_S \frac{p(\xi)\,d\xi}{x-\xi} - \beta q(x), \quad x \in S \tag{4}$$

where the integral has to be interpreted as the Cauchy principal value, and appropriate side conditions must be given to choose the physically meaningful solution from the space of the mathematical solutions, as discussed in the definitive treatise by Muskhelishvili (1953). These side conditions depends on the behaviour (bounded/unbounded) of the unknown functions at the ends of contact areas, and on whether the area is connected or not. Note that a solution to the integral equation itself, either in terms of displacements, or displacement derivatives, will not necessarily satisfy the inequality condition [eqn (3)], in particular in the case of multiply-connected contact area. This is generally checked *a posteriori*. In the previous equation

$$A = \frac{\kappa_1+1}{4\mu_1} + \frac{\kappa_2+1}{4\mu_2} \tag{5}$$

is a measure of the "composite compliance" of the bodies, and

$$\beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \tag{6}$$

gives a measure of the "elastic mismatch" and is known as a Dundurs' constant; moreover, κ is Kolosov's constant, given by $\kappa = (3-4\nu)$ under plane strain conditions (ν_i, μ_i are the Poisson's ratio and shear modulus of the material of body i). Equilibrium provides fixed side conditions: the resultant forces P (positive compressive force), tangential force Q, and moment M satisfy

$$P = -\int_S p(x)\,dx \tag{7}$$

$$Q = +\int_S q(x)\,dx \tag{8}$$

$$M = -\int_S p(x)x\,dx \tag{9}$$

where it is intended that M is the moment with respect to the origin of the coordinate system $x = 0$, and the contact area S is not necessarily connected.

The second integral equation defining the problem relates to displacement of particles parallel with the surface. It reads, again using displacement derivatives, as

$$\frac{1}{A}g'(x) = \frac{1}{\pi}\int_S \frac{q(\xi)\,d\xi}{x-\xi} + \beta p(x), \quad x \in S \tag{10}$$

where $g(x) = u_x(x)$ is the relative tangential displacement of surface particles, $g'(x) = dg(x)/dx$ its derivative, $q(x) = \sigma_{xy}$ (x, 0) the shearing traction distribution, and the integral has to be interpreted again as Cauchy principal value. Note that $h(x)$ and $g(x)$, whilst to some degree analogous, have very different roles in eqns (2) or (4), and eqn (10), as the first is prescribed whilst the second is, at this stage, a dependent variable. In fact, considering a pair of points in the two contacting bodies: before coming into contact, their relative tangential displacement is free (in particular, note that it can have either sign, depending on the elastic properties of the material; this differs from the condition for the normal contact (3), where the relative normal displacement of points not in contact *must* be less than the original gap between them, otherwise there is contact outside the intended domain). Also, $q(x)$ can have either sign, but $p(x)$ cannot. Therefore, whilst eqns (2) or (4) already incorporates a boundary condition, eqn (10) does not. When a pair of points comes into contact, the magnitude of their relative tangential displacement is still free, if they lie in a slip zone, providing its sign is consistent with the sign of the shear tractions (see below), whereas within the stick zone the relative tangential displacement of surface particles is equal to the value when the particles first enter the stick zone and, therefore†

$$g(x) = T_x + g_0(x), \quad x \in S_{\text{stick}}$$

or

$$g'(x) = g_0'(x), \quad x \in S_{\text{stick}} \tag{11}$$

where T_x is the tangential component of the rigid body motion, and $g_0(x)$ is the value before entering the stick area. For points outside the contact area $x \notin S$, eqn (10) still holds, as $g(x)$ is simply defined as the relative tangential displacement of pairs of surface particles.

The shearing traction must be less than the limiting value in S_{stick}, i.e.‡

$$|q(x)| < -fp(x), \quad x \in S_{\text{stick}}. \tag{12}$$

Further, within the slip zones the shearing traction is limited by friction, so that

† When a surface bulk strain, parallel with the surface is present, which is not due to the contact, it must be taken into account (Hills *et al.*, 1993).

‡ Note that the negative sign is due to our convention $p(x) = \sigma_{yy}$ $(x, 0)$.

$$|q(x)| = -fp(x), \quad x \in S_{\text{slip}} \tag{13}$$

and the shear traction must always oppose the direction of relative change in the direction of slip, i.e.

$$\operatorname{sgn}(q(x)) = \operatorname{sgn}\left(\frac{\partial g}{\partial t}\right), \quad x \in S_{\text{slip}}. \tag{14}$$

The above equations provide a framework for the solution of the problem, although they may not be exhaustive in the case of contact between elastically dissimilar components, for which an incremental formulation is required in general.

Considering the case when the contacting bodies are elastically similar, or more specifically, Dundurs' constant β vanishes†, i.e.

$$\frac{1-2\nu_1}{\mu_1} = \frac{1-2\nu_2}{\mu_2} \tag{15}$$

eqns (4) and (10) are uncoupled, and solutions for the normal and tangential loading may be obtained independently. For this general construction of the integral equations, it is clear that the equalities and inequalities are different in number and appearance, for normal and tangential displacements: there is no shear equivalent for inequality (3), and the conditions given by Coulomb's law (12)–(14) clearly have no equivalent for the normal direction.

However, let us consider a particular path in loading space, defined as follows: apply a normal force and hold it fixed while increasing a tangential force monotonically from zero. On applying the normal load, P, alone, there is no tendency in any case for surface particles to slip and, hence, the initial stick zone envelopes the entire contact, as

$$g_0'(x) = 0, \quad x \in S. \tag{16}$$

A monotonically increasing shearing force, Q, will then give rise to advancing slip, and under these circumstances, eqn (14) is automatically satisfied.

2.1. *Fully adhesive conditions*

At this stage, Cattaneo (1938) shows that, for Hertzian contacts, a fully adhesive solution implies a singular traction distribution at the edge of the contact area. In fact, this result is independent of the geometry of the indenter, as it follows immediately from substituting eqn (16) into eqn (10), obtaining

$$0 = \frac{1}{\pi}\int_S \frac{q(\xi)\,d\xi}{x-\xi}, \quad x \in S \tag{17}$$

and the solution of this equation is that of the normal contact problem for a rigid punch (with one, or several, flat areas of the same heights) pressed into the half plane and, therefore, is singular at each edge (Muskhelishvili, 1953). In particular, we give below the solution in the form derived by Scthayerman (1949). It may be appreciated that the solution will be *independent* of the actual solution of the contact problem, i.e. the form of shearing traction is the same for any profile, providing the contact area (or the distribution of contact areas) is the same.

2.2. *Partial slip conditions*

We now seek solutions for contacts with finite friction, where we have to assume that slip occurs at the edge of each contact area. In other words, the shearing traction distribution

† Note that the condition is valid not only for a couple of similar materials, but many other cases of engineering interest, like steel on rubber, satisfy the condition $\beta = 0$.

will be a correction of the full sliding term in the stick zone, of dimension and location presently unknown†. Also, it is possible to reconsider this idea from a different point of view. It is clear that, according to this classical construction of the integral equations, there is no symmetry between either the equalities or inequalities in the normal and subsequent tangential loading phases, even for elastically similar materials. Thus, it is not possible *a priori* to draw any particular general conclusion regarding the distribution of shearing traction in the stick area. Let us reconsider the problem in the partial slip regime obtained by Cattaneo's superposition of a full sliding component and a corrective part $q^*(x)$ in the stick zone, of location and dimension presently unknown, so that assuming a direction for Q such that the full sliding would be $Q < 0$, one has

$$q(x) = \begin{cases} fp(x) + q^*(x), & x \in S_{\text{stick}} \\ fp(x), & x \in S_{\text{slip}} \end{cases}. \tag{18}$$

Then, the integral equation for relative displacements in the tangential direction states, using eqn (16) again, and substituting eqn (4) for the full sliding component:

$$0 = \frac{1}{\pi}\int_S \frac{q(\xi)\,\mathrm{d}\xi}{x-\xi} = \frac{f}{A}h'(x) + \frac{1}{\pi}\int_{S_{\text{stick}}} \frac{q^*(\xi)\,\mathrm{d}\xi}{x-\xi}, \quad x \in S_{\text{stick}} \tag{19}$$

where S_{stick} is the stick zone, and $q^*(x) = 0$ in the slip zones, by definition. Then, $q^*(x)$ is the solution of the following integral equation:

$$\frac{1}{\pi}\int_{S_{\text{stick}}} \frac{-q^*(\xi)/f}{x-\xi}\mathrm{d}\xi = \frac{1}{A}h'(x), \quad x \in S_{\text{stick}} \tag{20}$$

which can be recognized as being of the same *form* as the original equation for normal contact for $\beta = 0$, eqn (4) [with $p(x)$ replaced by $-q^*(x)/f$, and the domain of the integral suitably scaled]. Note that the value of the function $h'(x)$ is given by the *actual* normal contact problem. In particular, it is clear that, if we had used displacements instead of displacement derivatives, the right-hand side of eqn (20) would differ by a quantity $-T_x/f$ with respect to $h(x)$, as can be seen by considering the value of $g(x)$ and not $g'(x)$ in eqn (11). In writing displacement derivatives, we lose information corresponding to the term $-T_x/f$, but we find a consistent solution, as usual, through the equilibrium side condition (in particular the load is lower, $|Q| < fP$). What is important, by contrast, is that we do keep information of the actual rotation in both the function $h(x)$ and $h'(x)$, and this gives the condition for the partial slip solution to be determined. Therefore, whereas in the normal loading phase it does not matter whether we reach the actual value of $h(x)$ through any particular path, as there is no path-dependence, in the analogy we need to consider the decrease of normal load, with a *fixed* value of rotation. In other words, whereas in the actual normal indentation, the resultant load P (considering the point of application for which there is no moment) can be applied following *any* path of application, in the analogy the load P has to be downloaded along a path that gives no change of rotation.

As in the normal contact problem, we force continuity of the shear traction (and displacements) throughout the stick/slip region, and require the corrective solution to be unbounded at the edges of the stick zone‡, in order to avoid singularities that are not allowed by Coulomb's law. Now, the analogy is only partly proved, as we need to check that the inequalities also correspond. Coulomb's law again requires $|q(x)| < -fp(x)$ in the stick zone [eqn (12)], and this maps into the condition that, according to the definition (18), $q^*(x)$ must be of opposite sign of $p(x)$, which is guaranteed by comparing eqn (4) with eqn (20), and also the requirement that contact tractions are compressive. The limiting

† To note that assuming finite friction does not mean that the shearing tractions are always a bounded function, as they may be singular if the normal pressure is singular.

‡ Except the limit case when the stick zone coincide with the entire contact area, that is possible in flat contacts, see below.

condition $|q(x)| = -fp(x)$ in the slip zones [eqn (13)] is implicitly satisfied by the assumption $q^*(x) = 0$ in the slip zones. Finally, the condition that the shear traction must always oppose the direction of relative change in the direction of slip [eqn (14)] is automatically satisfied by the proposed *rescaled* normal contact problem, as the condition of reverse slip starting in the stick zone is easily shown to be impossible and, therefore, the assumed sign of the full sliding component is correct in the entire slip zone, by comparing the resulting traction. This depends on a theorem in normal frictionless contact that the contact pressure at all points increases monotonically with the normal force, i.e. there is no point or situation at which an increment of normal force causes a decrease in local contact pressure (Shield, 1967; Barber, 1992, Section 21.2). Therefore, the condition of interpenetration [eqn (3)] also has an equivalent, which implies that, in general, relative tangential displacement is different from zero (in particular, the sign of relative displacements depends on the sign of the tangential force) in the slip zones, which simply means that wear has to result, and energy has to be dissipated in that region.

The inequalities that determine the size of the contact area during application of the normal load, correspond to inequalities that determines the size of the stick zone in the sequential tangential problem (i.e. the dimension of the stick zone is related to the applied load through the tangential equilibrium condition, eqn (8)).

The last remarks involve *side conditions*: for a simply-connected contact area, the side conditions for normal contact are:

- if the contact area is known *a priori*, the solution (pressure distribution) is unbounded at the edges, and equilibrium gives the only side condition necessary to find the arbitrary constant in the solution;
- if the contact area is unknown, two additional side conditions are provided by the requirement that the pressure distribution has to be bounded, and in particular zero, at the edges; these two conditions give the load, and the offset of the contact area. In the case of symmetry, one condition is clearly sufficient, and gives the load, as the offset is automatically determined to be zero with respect to the line of symmetry.

In both cases, if there is also rotation, the unknown angle of rotation (or the unknown moment) are determined from rotational equilibrium.

For a multiply-connected contact area (say N areas), similar conditions apply, namely:

- if the edges of the contact area are all known *a priori* (a set of flat surfaces), the solution (pressure distribution) is unbounded at the edges, equilibrium gives the side condition necessary to find one arbitrary constant in the solution, and $N-1$ other conditions are provided by the jumps in the relative normal displacements to be consistent with the jump in the profile, to give the other $N-1$ arbitrary constants in the solution;
- if some edges of the contact area are unknown *a priori*, a corresponding number of conditions are provided by the requirement that the pressure is zero at those edges.

These conditions *translate* without major modification into the corrective solution. In fact, for a single contact area, two situations are possible: the profile is non flat, and the corrective solution is zero at the boundaries of the stick zone; or the profile is flat, and then the corrective solution is of the same form as the full sliding one, equilibrium determining the single unknown arbitrary constant in the solution. For a set of flat areas, the contact is either in full sliding or full stick in each area, the condition being determined by the jump conditions, and equilibrium. For smooth profiles, the location of the stick zones is determined by the condition that the corrective solution is zero at that edges. The only difference is that the rotational equilibrium condition does not apply (the reason being that we have assumed Q to be applied at $y = 0$), but this is compensated by the fact that the rotation is fixed to the value in the actual contact problem; therefore, the number of conditions and variables is equal again.

The analogy between $p(x)$ in the normal contact problem in the contact area, and $-q^*(x)/f$ in the stick zone for tangential sequential loading is, therefore, fully and rigorously completed, and the theorems that ensure existence and uniqueness of the solution of the normal frictionless problem (Fichera, 1964; Duvaut and Lions, 1972), automatically prove existence and uniqueness of the corrective solution found for the partial slip problem. Also, they provide variational methods for the solution of the corrective problem (in particular the surface displacements and the true stick zone minimize the total strain energy of the corrective problem) (Kalker, 1977) in terms of minimization of the total complementary energy (Kikuchi and Oden, 1977). There is no need for us to restate these principles explicitly, as the corrective problem is mathematically exactly the same as a normal frictionless contact problem.

Therefore, the following properties of the corrective solution and the partial slip regime can be deduced immediately:

- if the indenter profile is symmetrical and self-similar, the corrective solution is of the same shape as the normal pressure in the contact area for any load;
- no partial slip solution can be predicted where the stick zone lies entirely within a flat region of the punch; in other words, flat regions are either entirely in full stick or are in full slip conditions. It is also possible to say that the local Coulomb law for friction, with a characteristic local stick–slip behaviour, becomes a global property for the flat part of the contact. In the limit case of an entirely flat contact, in particular a set of flat areas of equal height, the full stick–full slip behaviour is global, and the local Coulomb law translates into a global Coulomb law;
- if in normal indentation there is no change of relative rotation, then the points that come into contact last, are the first to slip; in general the path followed by points that lose contact during a monotonic decrease of the normal load from the actual value, but with a *fixed* rotation, gives the path of points entering slip zones during a monotonic application of tangential load from zero;
- if the indenter profile has discontinuities, these affect the tangential load, stick area dimension relation in the same way as they affect the normal load, contact area dimension. This will be clarified below.

Finally, the magnitude of relative shear displacement in the slip zones may be found directly from

$$g'(x) = \frac{1}{\pi}\int_S \frac{q(\xi)\,\mathrm{d}\xi}{x-\xi} = \frac{f}{A}h'(x) + \frac{1}{\pi}\int_{S_{\mathrm{stick}}} \frac{q^*(\xi)\,\mathrm{d}\xi}{x-\xi} < 0, \quad x \in S_{\mathrm{slip}} \tag{21}$$

which is an ordinary integral, much easier to compute numerically with any standard procedure (but it is very often possible to compute in closed form), and for which the inequality comes from eqn (3), as already discussed. The absolute displacements can be found by integrating from the stick zone, where $g'(x)$ is known.

3. SOLUTION OF THE GENERALIZED CATTANEO PROBLEM

Apart from these general properties, the analogy proved allows us to solve easily all the configurations for which the normal contact is solved. In the following, we give a full account of the case of a single contact area with arbitrary profile contact, where the solution is elementary, and can be found in closed form for many configurations, as discussed in details in part II of the paper. In more general cases, such as the multiple contact area, or periodic contact, the solutions for an arbitrary profile are quite cumbersome to use, and a direct numerical approach has generally been preferred in the past. Also, they are available often only in the Russian literature [as for example in Schtayerman monograph (1949)], but they give insight into properties of the contact itself. Moreover, the use of modern symbolic and numerical library software can justify their use in some cases, as this will certainly provide greater accuracy than a general direct numerical solution of the contact problem.

In particular, in Appendix 1 we recall the solution in the form given by Schtayerman (1949) for a general profile in multiple contact, over the range $a_i \leqslant x \leqslant b_i$, $i = 1, \ldots, n$. The solution depends on a set of coefficients $C_0, C_1, \ldots, C_{n-2}$ of a polynomial $P_{n-1}(x)$, that are determined to give to the vertical displacements $u_y(x)$ the proper jumps (the conditions are fully given in Appendix 1). If some contact edges are not known *a priori*, a corresponding number of equations is given from the condition that the pressure has to be zero there. If the number of contact areas is infinite, and the contact is periodical (the period does not necessarily involve only one contact area), the solution is again given by Schtayerman (1949) and reported in Appendix 2.

3.1. *Full stick condition—single contact area*

In the case of a single, simply-connected symmetrical contact region, the full adhesive solution for the shearing traction is

$$q(x) = -\frac{Q}{\pi\sqrt{a^2-x^2}} \tag{22}$$

and corresponds to the pressure distribution for a flat punch.

3.2. *Full stick conditions—multiple contact area*

The solution corresponds to the frictionless normal contact in the case of a profile composed of a set of flat surfaces, i.e. $h'(x) = 0$, for $a_m \leqslant x \leqslant b_m$, $m = 1, \ldots, n$, and in particular, for the case where the flat areas are at the same height, as can be understood from condition (11) for $g(x) = T_x$. Therefore, we find from the solution given in Appendix 1

$$q(x) = (-1)^{n-m}\frac{Q_{n-1}(x)}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \tag{23}$$

where

$$Q_{n-1}(x) = D_0 + D_1x + D_2x^2 +, \cdots, + D_{n-2}x^{n-2} - Qx^{n-1}. \tag{24}$$

Coefficients $D_0, D_1, \ldots, D_{n-2}$ of polynomial $Q_{n-1}(x)$ are determined from the system of linear equations in order to give to $g(x)$ the proper constant value, i.e. to satisfy the original integral equation in terms of displacements.

3.3. *Full stick conditions—periodic contact area*

The solution corresponds to the frictionless normal contact in the case of a periodical set of flat surfaces of equal height, i.e. $h'(x) = 0$, over the range $a_m \leqslant x \leqslant b_m$, $(m = 1, \ldots, n)$. Appendix 2 gives the general solution for the contact problem and, therefore, also of this particular case. In the case where the period is equal to only one contact area, which is perhaps the case of greater interest, $n = 1$, $h'(x) = 0$, over $l/2-a \leqslant x \leqslant l/2+a$, the solution is

$$q\left(\frac{l}{2}+x\right) = -\frac{Q\sqrt{2}\cos\frac{\pi x}{l}}{l\sqrt{\cos\frac{\pi x}{l}-\cos\frac{\pi a}{l}}}, \quad -a \leqslant x \leqslant a \tag{25}$$

where Q is the load per contact area. Note that in the limit when $l = 2a$, the shearing distribution goes towards the uniform limit.

3.4. *Partial slip conditions—simple contact area*

Starting again by recalling the results for the normal contact problem, let us consider the case of a single contact region, for the unsymmetrical function $h(x)$ case (the symmetrical case will be recovered easily). The size of the contact area is fixed by the normal load P. If the contact area is not symmetrical with respect to the origin, say $-a+\delta \leqslant x \leqslant a+\delta$, on assuming

$$t = \tau+\delta, \quad x = \xi+\delta \tag{26}$$

the integral equation becomes (Schtayerman, 1949)

$$\frac{1}{A}h'(\xi+\delta) = \frac{1}{\pi}\int_{-a}^{a}\frac{p(\tau+\delta)\,\mathrm{d}\tau}{\xi-\tau}, \quad -a \leqslant \xi \leqslant a \tag{27}$$

where a is the semi-dimension of the contact area, and δ is the offset with respect to the origin of coordinates $x = 0$, in which the function h is defined by eqn (1). The additional condition to determine the offset δ is provided by rotational equilibrium [eqn (9)]. The solution in the case of a general profile in contact over the range $-a \leqslant \xi \leqslant a$ is, under the hypothesis that both $p(\xi = a)$ and $p(\xi = -a)$ are bounded [the contact is called *incomplete* and it may be proved that in particular $p(\xi = \pm a) = 0$], is

$$p(\xi+\delta) = \frac{1}{\pi A}\sqrt{a^2-\xi^2}\int_{-a}^{a}\frac{h'(\tau+\delta)\,\mathrm{d}\tau}{\sqrt{a^2-\tau^2}(\tau-\xi)}, \quad -a \leqslant \xi \leqslant a \tag{28}$$

together with the conditions $p(\xi = \pm a) = 0$, and equilibrium between the applied load and the pressure distribution, translate into the following two equations

$$\int_{-a}^{a}\frac{h'(\tau+\delta)\,\mathrm{d}\tau}{\sqrt{a^2-\tau^2}} = 0, \quad P = -\int_{-a+\delta}^{a-\delta}p(t)\,\mathrm{d}t = -\frac{1}{A}\int_{-a}^{a}\frac{h'(\tau+\delta)\,\tau\,\mathrm{d}\tau}{\sqrt{a^2-\tau^2}}. \tag{29}$$

To determine the moment M, we require, for rotational equilibrium:

$$M = -\int_{-a+\delta}^{a+\delta}p(t)t\,\mathrm{d}t = -\int_{-a}^{a}p(\tau+\delta)(\tau+\delta)\,\mathrm{d}\tau$$
$$= P\delta-\int_{-a}^{a}p(\tau+\delta)\tau\,\mathrm{d}\tau = P\delta+\frac{1}{A}\int_{-a}^{a}h'(\tau+\delta)\sqrt{a^2-\tau^2}\,\mathrm{d}\tau. \tag{30}$$

Moving to the solution of the partial slip Cattaneo problem, as a consequence of the analogy established, the stick zone is not necessarily positioned symmetrically relative to the origin, i.e. $-c+\delta^* \leqslant x \leqslant c+\delta^*$, and so let us assume

$$t = \tau+\delta^*, \quad x = \eta+\delta^* \tag{31}$$

the corrective solution in the stick zone is

$$-q^*(\eta+\delta)/f = \frac{1}{\pi A}\sqrt{c^2-\eta^2}\int_{-c}^{c}\frac{h'(\tau+\delta)\,\mathrm{d}\tau}{\sqrt{c^2-\tau^2}(\tau-\eta)}, \quad -c \leqslant \eta \leqslant c \tag{32}$$

where c is the semi-dimension of the stick area, and δ^* is its offset with respect to the origin of coordinates $x = 0$, with respect to which the function h is defined as eqn (1). The conditions $q^*(\eta = \pm c) = 0$, and equilibrium translate into the following two equations

$$\int_{-c}^{c} \frac{h'(\tau+\delta^*)\,d\tau}{\sqrt{c^2-\tau^2}} = 0, \quad -Q^*/f = -\frac{1}{A}\int_{-c}^{c} \frac{h'(\tau+\delta^*)\,d\tau}{\sqrt{c^2-\tau^2}}. \tag{33}$$

These equations give the offset δ^*, and the size of the stick zone c. Note that, for a non-symmetrical self-similar profile (i.e. the functions on the right and the left of the $x = 0$ axes are each self-similar) the offset of contact area, and offset of stick zone are proportional, i.e. $\delta/\delta^* = c/a$, as the rotation is fixed. The tangential load may be calculated from

$$Q = fP + \frac{f}{A}\int_{-c}^{c} \frac{h'(\tau+\delta)\tau\,d\tau}{\sqrt{c^2-\tau^2}} \tag{34}$$

where c is the semi-dimension of the stick zone. Let us define

$$\Phi(x,y) = x\int_{-x}^{x} \frac{h'(t+y)t\,dt}{\sqrt{x^2-t^2}}. \tag{35}$$

The ratio of the transmitted forces Q/fP, which ranges from 0, for normal loading only contact, to 1, for full sliding, is, therefore, related to the stick zone size by

$$\frac{Q}{fP} = 1 - \frac{\Phi(c,\delta^*)}{\Phi(a,\delta)} \tag{36}$$

where the denominator is clearly constant upon increasing the tangential load, whereas the numerator gives the variation with c/a, and δ^*/δ. Note that the symmetrical case (and with no rotation) is recovered from the previous equations by setting $\delta^* = \delta = 0$.

3.5. *Partial slip conditions—multiple contact area*

The solution of the contact problem (i.e. in terms of the pressure), is given in Appendix 1 in the general form. On moving to the partial slip problem one has only to correct $h(x)$ by $-T_x/f$. However, as $h(x)$ appears only as its derivative $h'(x)$, or as a difference between edge values, there is no need to do this explicitly. Therefore, we need to just make use of the solution given in Appendix 1. First, define a set of stick areas $c_i \leqslant x \leqslant d_i$, $i = 1, \ldots, n$, and substitute a_i, b_i with $c_i, d_i, p(x)$ with $-q^*(x)/f$, the polynomial $P_{n-1}(x)$ with a polynomial $Q^*_{n-1}(x)$ with coefficients $D^*_0, D^*_1, \ldots, D^*_{n-2}$, and finally the load P with the load $-Q^*/f$.

3.6. *Partial slip conditions—periodical contact area*

The solution of the contact problem is given in Appendix 2 in the general form. On moving to the partial slip problem, appropriate substitutions should be made to find the dimension of the stick zones from the side conditions, and the form of the corrective shearing distribution in the stick zones. As the notation is quite heavy, we do not give explicit details of the resulting solution, as it is clear that the procedure follows the same rules of the preceding sections.

4. CONCLUSIONS

Partial slip contact has been considered and new general results developed, in the framework of contact of bodies that can be approximated as elastic half-planes. Cattaneo's problem of sequential tangential loading has been reworked, and we have shown that the corrective shearing distribution in the stick zone is given by the solution of a frictionless normal contact problem, for a lower load, but the same rotation of the actual one. This gives a general method of solution of particular cases, as fully exploited in part II of the paper.

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APPENDIX 1

The general solution of the multiple contact problem

It is not within the scope of the present paper to give a full account of all the configurations for which the normal contact is solved. However, we recall the general solution in quadrature for the multiple contact problem in order to permit a better understanding of the conditions that affect the pressure in the general case, and as these solutions are not easily available in the literature. Although they are quite cumbersome to write, it is believed that the availability of symbolic and numerical programs will permit their greater application. We rewrite the solution in the form given by Schtayerman (1949) for contact over the range $a_i \leqslant x \leqslant b_i$, $i = 1, \ldots, n$

$$p(x) = (-1)^{n-r} \frac{\dfrac{1}{A} \sum_{m=1}^{n} (-1)^{n-m} \displaystyle\int_{a_m}^{b_m} \frac{\sqrt{\left| \prod_{m=1}^{n} (t-a_m)(t-b_m) \right|}\, h'(t)\mathrm{d}t}{\tau - x} + P_{n-1}(x)}{\pi \sqrt{\left| \prod_{m=1}^{n} (x-a_m)(x-b_m) \right|}} \tag{A1}$$

where

$$P_{n-1}(x) = C_0 + C_1 x + C_2 x^2 + , \cdots, + C_{n-2} x^{n-2} - P x^{n-1}. \tag{A2}$$

Coefficients $C_0, C_1, \ldots, C_{n-2}$ of polynomial $P_{n-1}(x)$ are determined from the system of linear equations in order to give to $u_i(x)$ the proper jumps, i.e. to satisfy the original integral equation in terms of displacements, as discussed previously. In particular, it can be proved that they are expressible as

$$\sum_{l=0}^{n-2} C_l \int_{b_m}^{a_{m+1}} \frac{x^l}{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}} = (-1)^{n-m}\frac{\pi}{A}[h(a_{m+1})-h(b_m)] + P\int_{b_m}^{a_{m+1}} \frac{x^{n-1}}{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}} +$$

$$-\frac{\pi}{mA}\int_{b_m}^{a_{m+1}} \frac{\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\frac{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\,h'(t)\,\mathrm{d}t}{t-x}}{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}}\mathrm{d}x \quad \text{(A3)}$$

for $m = 1, \ldots, n-1$. If the contact edges are determined *a priori*, the conditions are sufficient to solve the problem; if, instead, some contact edges are to be determined, a equal number of equations will be determined from the condition that the presure has to be zero there. If for example, the pressure is bounded at edges $x = a_l, b_l$, by requiring the numerator of the solution for $p(x)$ to be zero at $x = a_l, b_l$, we get

$$\frac{1}{n}\frac{\pi}{A}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\frac{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\,h'(t)\,\mathrm{d}t}{t-a_l} + P_{n-1}(a_l) = 0 \quad \text{(A4)}$$

$$\frac{1}{n}\frac{\pi}{A}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\frac{\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\,h'(t)\,\mathrm{d}t}{t-b_l} + P_n(b_l) = 0. \quad \text{(A5)}$$

In the particular case of a profile composed by a set of flat surfaces, i.e. $h'(x) = 0$, for $a_m \leqslant x \leqslant b_m$, $m = 1, \ldots, n$, the solution for the pressure reduces to

$$p(x) = (-1)^{n-m}\frac{P_{n-1}(x)}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}. \quad \text{(A6)}$$

APPENDIX 2

The general solution of the periodic contact problem

The solution is available for the very general case of a periodic contact defined by an infinite number of contacts, with a periodicity of n contact areas per period, in the general case of profile, although it is quite cumbersome to write. We rewrite the Schtayerman solution (1949) for contact of period l, over the range $a_m \leqslant x \leqslant b_m$, $m = 1, \ldots, n$

$$p(x) = \frac{1}{Al}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-b_m)}{\sin\frac{\pi}{l}(x-a_m)}\right|}\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-a_m)}{\sin\frac{\pi}{l}(x-b_m)}\right|}\,h'(t)\cot\frac{\pi}{l}(t-x)\,\mathrm{d}t$$

$$+(-1)^{n-m}\frac{\sum_{m=1}^{n}\gamma_m\sin^{n-m}\frac{\pi x}{l}\cos^{m}\frac{\pi x}{l}}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(x-a_m)\sin\frac{\pi}{l}(x-b_m)\right|}} \quad \text{(A7)}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_n$ are determined by equations

$$-\gamma_0+\gamma_2-\gamma_4\ldots = \frac{P}{l}\sin\sum_{m=1}^{n}\frac{\pi}{2l}(a_m+b_m) + \frac{1}{Al}\left(\sin\sum_{m=1}^{n}\frac{\pi}{l}b_m\right)\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-a_m)}{\sin\frac{\pi}{l}(x-b_m)}\right|}\,h'(t)\,\mathrm{d}t \quad \text{(A8)}$$

$$-\gamma_1+\gamma_3-\gamma_5\ldots = \frac{P}{l}\cos\sum_{m=1}^{n}\frac{\pi}{2l}(a_m+b_m) + \frac{1}{Al}\left(\cos\sum_{m=1}^{n}\frac{\pi}{l}b_m\right)\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-a_m)}{\sin\frac{\pi}{l}(x-b_m)}\right|}\,h'(t)\,\mathrm{d}t \quad \text{(A9)}$$

and

$$\sum_{m=0}^{n} \gamma_m \int_{b_k}^{a_{k+1}} \frac{\sin^{n-m}\frac{\pi x}{l}\cos^{m}\frac{\pi x}{l}\,\mathrm{d}x}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(x-a_m)\sin\frac{\pi}{l}(x-b_m)\right|}} = \frac{(-1)^{n-k}}{A}\left[h(a_{k+1})-h(b_k)+\frac{1}{l}\int_{b_k}^{a_{k+1}}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-b_m)}{\sin\frac{\pi}{l}(x-a_m)}\right|}\right.$$

$$\left.\times\left(\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-a_m)}{\sin\frac{\pi}{l}(x-b_m)}\right|}\,h'(t)\cot\frac{\pi}{l}(t-x)\,\mathrm{d}t\right)\mathrm{d}x\right]$$

$$k = 1,2,\ldots,n-1. \quad \text{(A10)}$$

If some contact edges are to be determined, a equal number of equations will be determined from the condition that the pressure has to be zero there. For the pressure to be bounded at edges $x = a_m, b_m$ $(m = 1,\ldots,n)$, the following conditions have to be satisfied

$$\sum_{m=0}^{n} \gamma_m \sin^{n-m}\frac{\pi b_k}{l}\cos^{m}\frac{\pi b_k}{l} = 0 \quad \text{(A11)}$$

$$\sum_{m=0}^{n} \gamma_m \sin^{n-m}\frac{\pi a_k}{l}\cos^{m}\frac{\pi a_k}{l} = \frac{(-1)^{n-k}}{Al}\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(a_k-b_k)\right|\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-a_m)}{\sin\frac{\pi}{l}(x-b_m)}\right|}\,h'(t)\cot\frac{\pi}{l}(t-a_k)\,\mathrm{d}t$$

$$k = 1,2,\ldots,n-1. \quad \text{(A12)}$$

In the latter equations, if some boundaries are known *a priori*, the appropriate conditions must be dropped out.